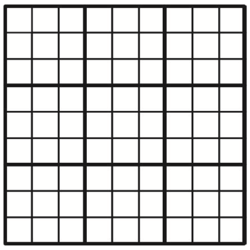
Solutions to Maths workbook - 2 | Permutation & Combination

Level – 3	Daily Tutorial Sheet – 14

228. $(n!)^{2n}$

Choose the first cell from first row in n^2 ways. Choose the second cell from the second row in $(n^2 - n)$ ways.



Similarly choose the third cell from the third row in $(n^2 - 2n)$ ways.

Proceed in the same fashion to select one cell from the nth row in $(n^2 - (n-1)n) = n$ ways. This procedure would have filled n rows, n columns and n sub grids.

Therefore, the total number of cells available for $(n+1)^{th}$ row in $(n^2 - n)$, for the $(n+2)^{th}$ $(n^2 - 2n)$ and so on.

Therefore, the total number of ways in $(n^2(n^2 - n)(n^2 - 2n)...(n))((n^2 - n)(n^2 - 2n)...(n))$ ... (1)

That we have found a solution using an elementary technique of counting, the result can also be verified by involving principle of Mathematical Induction. Try! While we are discussing PMI, is it also possible to invoke recurrence relation (that is in search of an elegant solution)?

229. $\dfrac{{}^{2n}C_n}{n+1}$

There are n 1's and n $2's$ in every arrangement. Total number of ways in which they can be arranged is $\dfrac{(2n)!}{n!n!}$ if there is no constraint.

Consider any permutation of $(n+1)2's$ and $(n-1)\ 1's$

All such permutations will be unbalanced. The total number of such permutations is

$\dfrac{(2n)!}{(n-1)!\,(n+1)!}$

Consider any permutation of n 1's and n 2's that is not balanced.

Now call i^{th} position as the critical position where the number of 2's exceeds the number of 1's for the first time. Therefore, before the i^{th} position, the arrangement is balanced and to the right of the i^{th} position the number of 1's exceed the number of 2's by 1. Now to the right of the critical position, change every 1 by 2 and every 2 by 1. This is one of the permutations of (n + 1) 2's and $(n-1)\ 1's$. In fact, there's a bijection between the total number of unbalanced arrangements and total number of arrangements of (n + 1)2's and (n – 1) 1's. (why?)

Therefore, total number of balanced arrangements $= \dfrac{2n!}{n!n!} - \dfrac{2n!}{(n+1)!\,(n-1)!} = \dfrac{1}{n+1}\ {}^{2n}C_n$.

230. $\frac{1}{2}6^n - (-3)^n$

Let a_n be the total number of students at the end of n years. Then,

$a_n = 3a_{n-1} + 18a_{n-2}$

Given $a_1 = 6$ and $a_2 = 9$

$a_n = 3a_{n-1} + 18a_{n-2} \quad \Rightarrow \quad a_n + 3a_n - 1 = 6(a_{n-1} + 3a_{n-2})$

Let $a_n + 3a_{n-1} = bn$. Then, $\boxed{b_n = 6b_{n-1}}$ for $n \geq 3$.

Clearly the sequence (b_n) is a G.P. with 6 as its common ratio.

Now, $b_2 = a_2 + 3a_1 = 27$

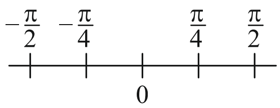
Obviously $b_n = 27 \cdot (6)^{n-2}$

$a_n + 3a_{n-1} = 27 \cdot (6)^{n-2} = \frac{3}{4}(6)^n$

Similarly, it can be shown $a_n = \frac{1}{2}(6)^n - (-3)^n$ (How?)

231. Given a real number t, we can find a unique real number t^* lying strictly between $-\pi/2$ and $\pi/2$ such that $\tan t^* = t$. (this is possible because the tangent function is strictly increasing on $]-\pi/2, \pi/2[$ and has range R). Therefore, corresponding to the five given real numbers, we can find five distinct real numbers, all lying between $-\pi/2$ and $\pi/2$.

Divide the open intervals $]-\pi/2, \pi/2[$ in to four equal intervals, each of length $\pi/4$



By the pigeon hold principle at least two of them must lie in some one of these intervals.

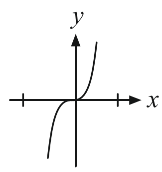
Suppose $x^*, y^* (x^* > y^*, \text{say})$ lie in the same interval.

Then $0 < x^* - y^* < (\pi/4)$.

Since the tangent function is strictly increasing, therefore

$$0 < \tan(x^* - y^*) < \tan(\pi/4)$$

i.e., $$0 < \frac{\tan x^* - \tan y^*}{1 + \tan x^* \tan y^*} < 1,$$



or $$0 < \frac{x - y}{1 + xy} < 1,$$

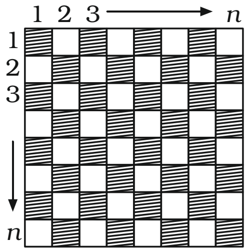
since $\tan x^* = x, \tan y^* = y$.

Remark. The graph of the tangent function is as in figure. 4.8

232. Divide the unit square into n^2 squares of edge length $\frac{1}{n}$ each. Now, using P.H.P. there should be least one square which will have two points. The length of diagonal of such a square will be $\frac{\sqrt{2}}{n}$. Therefore, there will be at least two points such that the distance between them does NOT exceed $\frac{\sqrt{2}}{n}$.

233.(8) Note: In this question n > 2.

We shall call the cell in i^{th} row, j^{th} column as $(i, j)^{th}$ cell.
Each cell occurs in exactly one row and one column. Therefore, each of the n^2 numbers will form Part of two AP's.



The number 1 must be the first (or lest!) term of each of the two A.P.'s to which it belongs. Therefore, it must be one of the four corner cells. There are thus four ways of placing 1 in a suitable cell.
Suppose 1 is placed in (1,1)th cell. Having fixed up the position of 1, let us see where we should place 2. The only choices for 2 are (1,2)th cell or (2,1)th cell. For, suppose we put 2 in (i,j)th cell, $i \neq 1, 2, j \neq 1, 2$. Then there must be $(i-1)$ numbers preceding it in jth column, and $(j-1)$ numbers preceding it in ith column. This is not possible since there is only one number, namely 1, less than 2. And it is also easy to rule out putting 2 in the remaining 3 corner cells. Certainly, 2 can't be placed at the other corner of the same row and column as 1 as there can't be $(n-2)$ numbers between 1 and 2.
Also, if 2 is placed in the opposite corner, then let (1, n) square have some $k = (1 + (n-1)d)$ where d is the common difference of the AP in 1st row. Then $1 + (n-1)\, d + (n-1)\, d_1 = 2$ where d_1 is common difference of the AP in last column. But $(n-1)\left[d + d_1\right] = 1$ isn't possible as $n > 2$.
When 1 occupies (1,1)th cell, there are two choices for 2. Similarly, for each of the other three possible ways of placing 1 in a cell, there are two ways of placing 2 in a cell, therefore there are in all $8(i.e., 4 \times 2)$ ways of placing 1 and 2 in the cells.
We shall now show that once the positions of 1 and 2 are fixed, the remaining numbers must be placed in the remaining $n^2 - 2$ cells is exactly one way (so as to satisfy the given conditions)
Suppose 2 is placed in (1,2)th cell. Then the first row must consist of the number 1, 2, 3,…,n
Let us now see as to where possibly we can place n^2. Since it is the largest of all the numbers, it must be the last (or first!) element of both the A.P.'s to which it belongs. Therefore, it must be in one of the corner cells, i.e., either in $(n,1)th$ cell or $(n,n)th$ cell. Suppose n^2 is placed in the $(n,1)th$ cell. Then the common difference of the A.P. formed by numbers in the first column must be $n+1$, so that the first common must read $1, n+2, 2n+3, 3n+4, \ldots, n^2$. Where do we place $n+1$?. It can be easily seen that since $n, n+2$ are placed far apart, it is not possible to place $n+1$ anywhere. Consequently n^2 cannot be the $(n,\, 1)th$ place. Suppose n^2 is placed in the $(n,n)th$ place. The last column should consist of the numbers $n, 2n, 3n, \ldots, n^2$. Since the numbers in the second row must be in A.P., with $2n$ as the last term, the only choice for 2nd row is $n+1, n+2, \ldots, 2n$ once this is done, we find that the first column must consist of the numbers $1, n+1, \ldots, n^2 - n + 1$, the third row must consist of the numbers $2n+1, \ldots, 3n$; the fourth row must consist of the numbers $3n+1, .., 4n$; and so on. This fixes the positions of all the n^2 numbers.

Since there are 8 ways of placing the numbers 1, 2 in two cells, and for each of these ways there is exactly one way of placing the remaining numbers, there are in all 8 ways of placing all the numbers in the various cells.